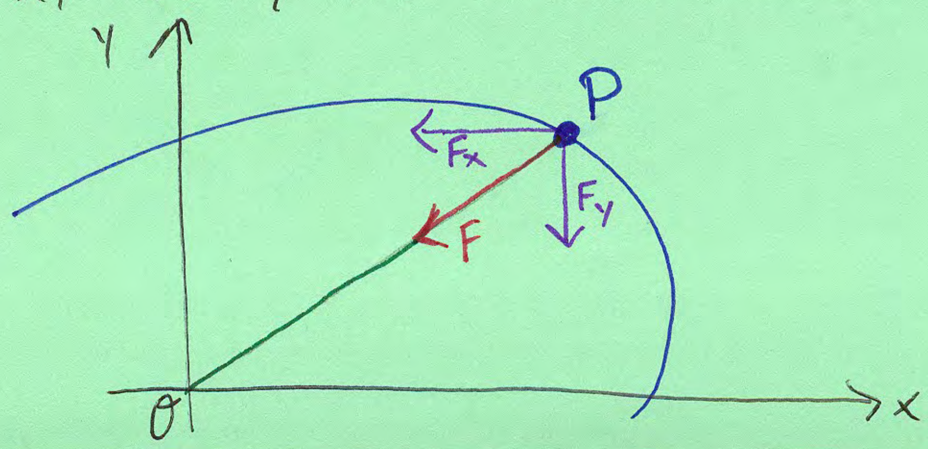
Centripetal Force in Cartesian Coordinates

Let P be a point-mass with mass m, moving in a plane, acted upon by a centripetal force with center the origin of the plane.

Let $x = x(t)$ and $y = y(t)$ be the coordinates of P as functions of time, $t \geq 0$. The magnitude of the force can also vary with time, so we write $F = F(t)$. We assume that the motion of P is smooth enough so that $x(t), y(t)$, are twice (at least) differentiable and that $F(t)$ is differentiable. Let r be the distance from O to P and let F_x and F_y be the components of F in the x & y directions, resp. Then r, F_x, and F_y are differentiable.



Using properties of similar triangles:

$$\frac{|F_x|}{F} = \frac{|x|}{r} \quad \& \quad \frac{|F_y|}{F} = \frac{|y|}{r}$$

Notice that the signs of F_x & x and F_y & y are always opposite, so that

$$\frac{F_x}{F} = \frac{-x}{r} \quad \& \quad \frac{F_y}{F} = \frac{-y}{r}$$

$$\Rightarrow rF_x = -xF \quad \& \quad rF_y = -yF$$

Now, Newton's second law gives:

$$F_x = m\frac{d^2x}{dt^2} \quad \& \quad F_y = m\frac{d^2y}{dt^2}$$

Thus, $mr\dfrac{d^2x}{dt^2} = -xF$ & $mr\dfrac{d^2y}{dt^2} = -yF$

Subtracting y times the first & x times the second:

$$mry\frac{d^2x}{dt^2} - mrx\frac{d^2y}{dt^2} = mr\left(y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2}\right) = -yxF + -xyF = 0$$

This gives: $$y\frac{d^2x}{dt^2} = x\frac{d^2y}{dt^2} \quad (*)$$

We are interested in the quantity $x\frac{dy}{dt} - y\frac{dx}{dt}$ (it relates to $\frac{d\theta}{dt}$). Observe

$$\frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \cancel{\frac{dx}{dt}\frac{dy}{dt}} + x\frac{d^2y}{dt^2} - \cancel{\frac{dy}{dt}\frac{dx}{dt}} - y\frac{d^2x}{dt^2}$$

$$= x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0 \quad \text{by } (*)$$

Therefore, $$\boxed{x\frac{dy}{dt} - y\frac{dx}{dt} = c}$$

This relation tells us important information about the motion of P which we can decode by going to polar coordinates!

Let $r = f(\theta)$ be a function describing the motion of the particle P. Notice that $\theta = \theta(t)$ & $r = r(t) = f(\theta(t))$. By our smoothness assumptions, $\theta, r,$ & f are twice differentiable.

We assume $t=0$ is an instant where P crosses the polar axis so that $\theta(0) = 0$. Take $\theta(t) > 0$ for $t > 0$ & $\theta(t) < 0$ for $t < 0$.

Now, let's change $x\frac{dy}{dt} - y\frac{dx}{dt} = c$ to polar:

$$x\frac{dy}{dt} = r\cos\theta \frac{d}{dt}(r\sin\theta) = r\cos\theta\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)$$

$$= \cancel{\frac{dr}{dt} r\cos\theta\sin\theta} + r^2\cos^2\theta\frac{d\theta}{dt}$$

$$y\frac{dx}{dt} = r\sin\theta\frac{d}{dt}(r\cos\theta) = r\sin\theta\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)$$

$$= \cancel{\frac{dr}{dt} r\sin\theta\cos\theta} - r^2\sin^2\theta\frac{d\theta}{dt}$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\cos^2\theta\frac{d\theta}{dt} + r^2\sin^2\theta\frac{d\theta}{dt} = r^2\frac{d\theta}{dt}$$

So, we get

$$r^2 \frac{d\theta}{dt} = \boxed{(r(t))^2 \theta'(t) = c}$$

Now, consider P at two times t_0 & t_1 $(t_0 \leq t_1)$ and let A be the area swept out in this time by OP. Let $\theta(t_0) = a$ & $\theta(t_1) = b$, so

$$A = \int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta$$

Recall $r(t) = f(\theta(t))$. In terms of t:

$$d\theta = d(\theta(t)) = \theta'(t)\, dt$$

Thus

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_{t_0}^{t_1} \frac{1}{2} r(t)^2 \theta'(t)\, dt$$

$$= \int_{t_0}^{t_1} \frac{c}{2}\, dt = \frac{1}{2} c(t_1 - t_0)$$

Let $k = \frac{1}{2}c$, then

$$\boxed{A = k(t_1 - t_0)}$$

So, the area swept out is equal to K times the time it takes to sweep it out!

Which law have we just verified?

Kepler's Second Law!

We call K the Kepler constant of the orbit.

Now we turn our focus to converting

$$mr\frac{d^2x}{dt^2} = -F_x \;\&\; mr\frac{d^2y}{dt^2} = -F_y \quad (**)$$

to polar.

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\cos\theta - r\sin\theta\cdot\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}$$

$$= \frac{d^2r}{dt^2}\cos\theta - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - \sin\theta\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)$$

Because $(r(t))^2\theta'(t)=2k$,

$$0=\frac{d}{dt}(2k)=\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=2r\frac{dr}{dt}\frac{d\theta}{dt}+r^2\frac{d^2\theta}{dt^2}$$

So, we have $\dfrac{d^2x}{dt^2}=\cos\theta\left[\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2\right]$

Similarly,
$$\frac{d^2y}{dt^2}=\sin\theta\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]$$

Putting these into (**) gives:

$$F\cos\theta=m\cos\theta\left[r\left(\frac{d\theta}{dt}\right)^2-\frac{d^2r}{dt^2}\right]\quad\text{and}$$

$$F\sin\theta=m\sin\theta\left[r\left(\frac{d\theta}{dt}\right)^2-\frac{d^2r}{dt^2}\right]$$

(after cancelling the r's on both sides)

Since $\sin\theta$ & $\cos\theta$ are not simultaneously zero:

$$F=m\left[r\left(\frac{d\theta}{dt}\right)^2-\frac{d^2r}{dt^2}\right]$$